# A New Discrete Impulse Response Gramian and its Application to Model Reduction

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***Abstract*—Some fundamental properties of a new impulse response Gramian for linear, time-invariant, asymptotically stable, discrete single-input–single-output (SISO) systems are derived in this note. This Gramian is system invariant and can be found by solving a Lyapunov equation. The connection with standard controllability, observability, and cross Gramians is proven. The significance of these results in model-order reduction is highlighted with an efficient procedure.**

***Index Terms*—Discrete time systems, Gramians, model reduction.**

## I. Introduction

Impulse response Gramians (IRG) have been introduced by Sreeram and Agathoklis to derive reduced-order models (ROM) for linear, time-invariant, asymptotically stable, continuous [1] or discrete [2] single-input–single-output (SISO) systems. Usefulness of these Gramians has also been shown in a system identification application.

An IRG contains elements that are inner products of functions given as successive derivatives or delays of the impulse response in continuous case and discrete case, respectively. It can be obtained by solving the Lyapunov equation for the controllability canonical realization of the system.

In [1], the approach is based on matching the first $q$ Markov parameters and $q \times q$ entries of the IRG. The procedure has been extended to the discrete case in [3], where the relation to the $q$-Markov cover method is discussed [4]–[6]. This method usually yields good approximations at high frequencies, but a large error on the steady-state behavior is noticed. An improved low-frequency approximation is achieved for discrete systems in [10] by matching some initial time moments and low-frequency power moments. For continuous systems, this drawback has been overcome with a reciprocal transformation [7], [8] to preserve the first $q$ time moments and $q \times q$ entries of the Gram matrix [9]. Another ROM building procedure, both valid in the continuous case [11] and in the discrete case [2], is based on the approximation of a energy criteria by a diagonalization and a direct truncation of the IRG. Use of the singular perturbation technique is suggested if a good approximation at low frequency is required. Note that the methods in [1], [2], and [11] still apply if the IRG is weighted (WIRG).

The approach in [2] has been recently extended to multiple-input–multiple-output (MIMO) systems with the definition of an extended impulse response Gramian (EIRG), and a convergence property to balanced realization [13], [14] has been established.

Krajewski *et al.* have proposed a mixed use of the results in [1] and [8] to derive a ROM matching Markov parameters, time moments, and impulse response energies [15]. It is based on a generalized definition of the IRG of Sreeram and Agathoklis using successive derivatives or integrals of the impulse response. This method is efficient but applies only to continuous time systems.

The initial motivation for the present paper is the extension of this approach to discrete case. A generalized impulse response Gramian (GIRG) composed with scalar products of successive differences or sums of the impulse response is introduced for linear, time-invariant, asymptotically stable, discrete SISO systems. It is related to standard controllability, observability, and cross Gramians and is found to be the solution to the Lyapunov equation for a particular state-space representation. It is also shown that the characteristic polynomial can be obtained using some impulse energies contained in the GIRG. Application of these properties to model reduction is then investigated, and an efficient procedure is proposed. The ROM is elaborated in two major steps: a reduced characteristic polynomial is first computed and then some Markov parameters or time moments are retained. The stability and minimality properties of this ROM are studied. A numerical example is proposed, and a comparison with well-known discrete model reduction techniques is carried out.

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## II. The Discrete Generalized Impulse Response Gramian

In this section, we first define the GIRG and then describe properties of this Gramian.

Let $(A, b, c)$ be an $n$th-order, minimal, state-space realization of a stable, linear, discrete SISO system with impulse response $h[k] = cA^{k-1}b$.

*Definition 2.1:* The $(n+1)$th-order GIRG is defined as follows:

$$W_{q,n+1} \hat{=} [\langle w_{q+i-1}, w_{q+j-1}\rangle]_{i,j=1,\cdots,n+1}, \quad q = -n+1, \cdots, 0, \tag{1}$$

with $w_0[k] \hat{=} h[k]$ and

$$\begin{aligned} w_{l+1}[k] &\hat{=} w_l[k+1] - w_l[k], \qquad l \in \mathbb{N}^+; \\ w_{l-1}[k] &\hat{=} -\sum_{l'=k}^{\infty} w_l[l'], \qquad l \in \mathbb{N}^- \end{aligned} \tag{2}$$

and where $\langle f, g\rangle \hat{=} \sum_{k=1}^{\infty} f[k]g[k]$ denotes the inner product of two causal real functions $f[k]$, $g[k]$.

Successive differences and sums of the impulse response defined in (2) have been previously used as candidates for constructing a set of approximating functions in [16]. An interesting property of such operators is that they preserve the original poles in the $z$-domain.

Some key properties of the GIRG are now considered in the following theorem.

*Theorem 2.1:*

i) The $n$th-order GIRG $W_{q,n}$ can be written as

$$\begin{aligned} W_{q,n} &= \mathcal{C}_q^T W_o \mathcal{C}_q \\ W_{q,n} &= \mathcal{O}_q W_c \mathcal{O}_q^T \\ W_{q,n} &= \mathcal{O}_q W_{co} \mathcal{C}_q \end{aligned} \tag{3}$$

where $W_c$, $W_o$, and $W_{co}$ denote, respectively, the standard controllability, observability, and cross Gramian for any minimal realization $(A, b, c)$.

The matrices $\{\mathcal{C}_q, \mathcal{O}_q\}$ used in the above factorizations are given by

$$\mathcal{C}_q = \left[(A-I)^q b, \cdots, (A-I)^{q+n-1} b\right]$$

and

$$\mathcal{O}_q^T = \left[[c(A-I)^q]^T, \cdots, [c(A-I)^{q+n-1}]^T\right]. \tag{4}$$

ii) $W_{q,n}$ is the solution to the Lyapunov equation

$$W_{q,n} - \hat{A}^T W_{q,n} \hat{A} = \hat{c}_q^T \hat{c}_q \tag{5}$$

where $(\hat{A}, \hat{b}_q, \hat{c}_q)$ is derived from $(A, b, c)$ by the coordinate transformation $\mathcal{C}_q$.

iii) The realization $(\hat{A}, \hat{b}_q, \hat{c}_q)$ has the following structure [in accordance with the proposed values for $q$; see (1)]:

$$\hat{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\overline{a}_n \\ 1 & 1 & \ddots & \vdots & -\overline{a}_{n-1} \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & -\overline{a}_2 \\ 0 & \cdots & 0 & 1 & -\overline{a}_1 + 1 \end{bmatrix} \tag{6}$$

$$\hat{b}_q^T = \Big[\underbrace{0, \cdots, 0}_{-q}, 1, \underbrace{0, \cdots, 0}_{q+n-1}\Big]$$

$$\hat{c}_q = \Big[\underbrace{\cdots, -t_2, -t_1}_{-q}, \underbrace{m_1', m_2', \cdots}_{q+1}\Big] \tag{7}$$

where $\{t_i\}_{i=1,2,\cdots}$ are the time moments of the system and $\{m_i' = c(A-I)^{i-1}b\}_{i=1,2,\cdots}$ are given as linear combinations of the Markov parameters $\{m_i\}_{i=1,2,\cdots}$

$$\begin{aligned} t_i &= c(I-A)^{-i}b \\ m_i &= cA^{i-1}b \\ m_i' &= \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} m_{i-j} \end{aligned} \tag{8}$$

and where $\{\overline{a}_i\}_{i=1,\cdots,n}$ denote the characteristic polynomial coefficients for $(A-I)$.

*Proof:* i) Starting with $w_0[k] = h[k]$, it is easily shown that, for any $l \in \mathbb{N}$, the function $w_l[k]$ derived using one of the transformations in (2) can be expressed as $w_l[k] = c(A-I)^l A^{k-1} b = cA^{k-1}(A-I)^l b$.

Writing each inner product $\langle w_{q+i-1}, w_{q+j-1}\rangle$ appearing in (1) by the $(A, b, c)$ matrices then yields directly the relations in (3). As $A$ is assumed to be asymptotically stable ($\|A\| < 1$), the matrix $(A-I)$ is nonsingular [19]. Thus, the existence of $\{\mathcal{C}_q, \mathcal{O}_q\}$ is ensured, and because $\{A, b\}$ is controllable, $\mathcal{C}_q$ is nonsingular.

ii) The observability Gramian for $\{\hat{A}, \hat{c}_q\}$ is given by $\mathcal{C}_q^T W_o \mathcal{C}_q$, which is seen to be the $n$th-order GIRG for $h[k]$ in view of (3).

iii) Let $q = 0$ and $\overline{p}(z) = \sum_{i=0}^n \overline{a}_i z^{n-i}$ be the characteristic polynomial for $(A-I)$. It is well known that a similarity transformation using the standard controllability matrix yields a state matrix under companion form

$$\mathcal{C}^{-1}A\mathcal{C} = \hat{A}' = \left[\begin{array}{c|c} \mathbf{0} & \\ \hline I_{n-1} & -\boldsymbol{a} \end{array}\right], \qquad \mathcal{C} = [b \;\; Ab \; \cdots \; A^{n-1}b]$$

where $\boldsymbol{a}^T = [a_n, \cdots, a_1]$ and $p(z) = \sum_{i=0}^n a_i z^{n-i}$ is the characteristic polynomial of $A$.

Then, it follows that

$$\mathcal{C}_0^{-1}(A-I)\mathcal{C}_0 = (\widehat{A-I})' = \left[\begin{array}{c|c} \mathbf{0} & \\ \hline I_{n-1} & -\overline{\boldsymbol{a}} \end{array}\right]$$

where $\mathcal{C}_0 = [b \;\; (A-I)b \; \cdots \; (A-I)^{n-1}b]$ and $\overline{\boldsymbol{a}}^T = [\overline{a}_n, \cdots, \overline{a}_1]$.

Then, we get (6). As the differences and sums in (2) preserve the original poles, the state matrix is the same for $q = -n+1, \cdots, 0$. The proof of (7) is straightforward and omitted. ■

*Remark 1:* As $\{\hat{A}, \hat{c}_q\}$ is observable, it follows from (5) that $W_{q,n}$ is positive definite and the $l$th-order GIRG $W_{q,l}$ is positive definite for any $l \leqslant n$.

The following theorem shows that the characteristic polynomial can be extracted from the GIRG.

*Theorem 2.2:* Let $p(z) = \sum_{i=0}^n a_i z^{n-i}$ ($a_0 = 1$) be the characteristic polynomial for any minimal realization $(A, b, c)$ of the system. Let the corresponding $(n+1)$th-order GIRG be partitioned as

$$W_{q,n+1} = \left[\begin{array}{c|c} W_{q,n} & \boldsymbol{w}_{q,n+1} \\ \hline \boldsymbol{w}_{q,n+1}^T & w_{q,n+1} \end{array}\right]$$

with $W_{q,n}$ the $n$th-order GIRG, $\boldsymbol{w}_{q,n+1} \in \mathbb{R}^{n\times 1}$, and $w_{q,n+1} \in \mathbb{R}$. Then, the following equation holds:

$$\overline{\boldsymbol{a}} = -(W_{q,n})^{-1}\boldsymbol{w}_{q,n+1} \tag{9}$$

where $\overline{\boldsymbol{a}}^T = [\overline{a}_n, \cdots, \overline{a}_1]$ and $\sum_{i=0}^n \overline{a}_i z^{n-i} = p(z+1)$.

*Proof:* By definition, $\boldsymbol{w}_{q,n+1}$ is given as

$$\boldsymbol{w}_{q,n+1} = \sum_{k=1}^{\infty} [w_q[k], \cdots, w_{q+n-1}[k]]^T w_{q+n}[k]$$

$$\text{with } w_{q+n}[k] = cA^{k-1}(A-I)^{q+n}b. \tag{10}$$

Let $\overline{p}(z) = \sum_{i=0}^n \overline{a}_i z^{n-i}$ be the characteristic polynomial for $(A-I)$. Then, from the Cayley–Hamilton theorem, we get

$$(A-I)^{q+n} = -\sum_{i=1}^n \overline{a}_i (A-I)^{q+n-i}$$

and

$$w_{q+n}[k] = -\sum_{i=1}^n \overline{a}_i w_{q+n-i}[k]. \tag{11}$$

Finally, substituting (11) into (10) yields (9). ■

Usefulness of these results in model-order reduction is shown in the next section.

## III. Model Order Reduction

Let us consider an $n$th-order original model described by the stable proper transfer function $H(z) = N(z)/D(z)$ with a minimal realization $\{x[k+1] = Ax[k] + bu[k], y[k] = cx[k]\}$.

The objective of model reduction is to find a state-space realization $\{\tilde{x}[k+1] = A_r\tilde{x}[k] + b_r u[k], \tilde{y}[k] = c_r\tilde{x}[k]\}$ with $\tilde{x}[k] \in \mathbb{R}^{r\times 1}$ and $r < n$, such that $\tilde{y}$ approximates $y$ as close as possible for all admissible inputs.

Let $W_{q,r+1}$ and $\tilde{W}_{q,r+1}$ be the $(r+1)$th-order GIRG for the original and reduced-order model, respectively, with the $q$ parameter chosen in the set $\{-r+1, \cdots, 0\}$.

An efficient GIRG-based model reduction technique will be proposed in the following. This technique can be seen as an extension to discrete systems of the approach considered in [15]. It consists of an approximation of some impulse response energies by first finding a reduced-order characteristic polynomial $\tilde{p}_q(z)$ and then matching some Markov parameters or time moments.

Let the Gramians $W_{q,r+1}$ and $\tilde{W}_{q,r+1}$ be partitioned as

$$\begin{aligned} W_{q,r+1} &= \left[\begin{array}{c|c} W_{q,r} & \boldsymbol{w}_{q,r+1} \\ \hline \boldsymbol{w}_{q,r+1}^T & w_{q,r+1} \end{array}\right] \\ \tilde{W}_{q,r+1} &= \left[\begin{array}{c|c} \tilde{W}_{q,r} & \tilde{\boldsymbol{w}}_{q,r+1} \\ \hline \tilde{\boldsymbol{w}}_{q,r+1}^T & \tilde{w}_{q,r+1} \end{array}\right]. \end{aligned} \tag{12}$$

Suppose that $\tilde{W}_{q,r}$ matches the original $r$th-order GIRG: $\tilde{W}_{q,r} = W_{q,r}$.

We will now calculate an $r$th-degree polynomial $\tilde{p}_q(z)$ such that $\|\tilde{\boldsymbol{w}}_{q,r+1} - \boldsymbol{w}_{q,r+1}\|_2^2$ is minimized. From Theorem 2.2, $\tilde{\boldsymbol{w}}_{q,r+1} = -W_{q,r}\tilde{\overline{\boldsymbol{a}}}_q$, where $\tilde{\overline{\boldsymbol{a}}}_q^T = [\tilde{\overline{a}}_r, \cdots, \tilde{\overline{a}}_1]$ and $\tilde{\overline{p}}_q(z) = \sum_{i=0}^r \tilde{\overline{a}}_i z^{r-i} = \tilde{p}_q(z+1)$. Hence, $\tilde{\boldsymbol{w}}_{q,r+1}$ matches $\boldsymbol{w}_{q,r+1}$ if

$$\tilde{\overline{\boldsymbol{a}}}_q = -(W_{q,r})^{-1}\boldsymbol{w}_{q,r+1}. \tag{13}$$

As $W_{q,r}$ is positive definite, its nonsingularity is ensured (Remark 1). Once $\tilde{\overline{\boldsymbol{a}}}_q$ has been computed, a reduced-order state matrix $\hat{\tilde{A}}_q$ with a

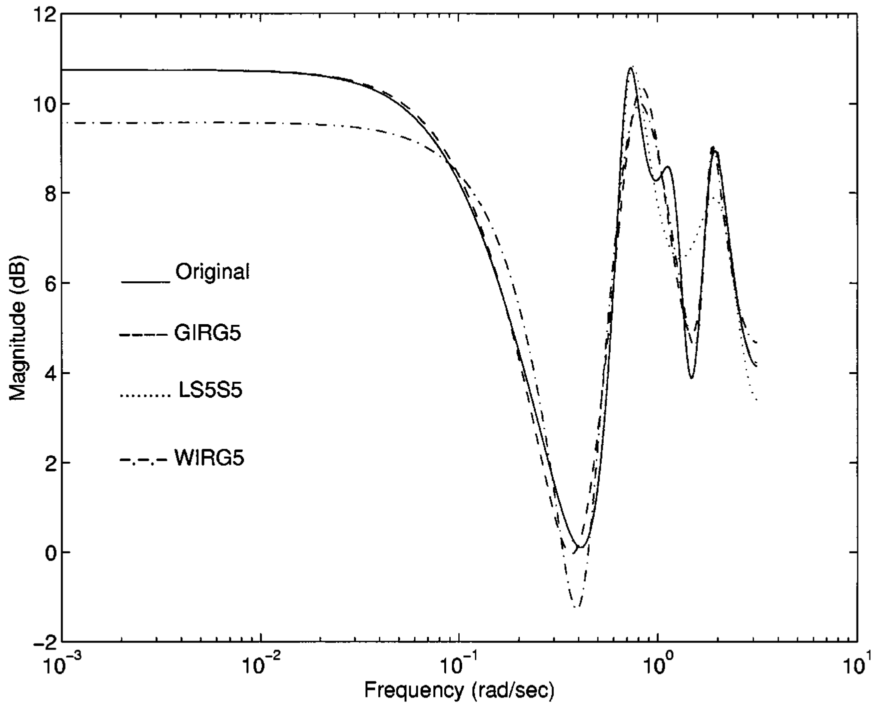


Fig. 1. Bode plots (magnitude).

form like (6) is readily obtained; it remains to choose input and output vectors $(\hat{\hat{b}}_q, \hat{\hat{c}}_q)$ to get our ROM.

From Theorem 2.1, we know that the $r$th-order GIRG for the ROM solves the Lyapunov equation

$$\tilde{W}_{q,r} - \hat{\hat{A}}_q^T \tilde{W}_{q,r} \hat{\hat{A}}_q = \hat{\hat{c}}_q^T \hat{\hat{c}}_q \tag{14}$$

where $\{\hat{\hat{b}}_q, \hat{\hat{c}}_q\}$ have a form like (7).

The matrices $\{\hat{\hat{A}}_q, \hat{\hat{b}}_q\}$ are known assuming that a characteristic polynomial for the ROM has been computed using (13). Then, (14) suggests that $\hat{\hat{c}}_q$ may be chosen so that some of the time moments or Markov-type parameters (8) of the original model are matched.

The main steps involved in our reduction procedure are then summarized as follows.

Step 1) Given an $n$th-order original model $(A, b, c)$, choose any $q \in \{-r+1, \cdots, 0\}$ with $r < n$ and then determine the particular realization $(\hat{A}, \hat{b}_q, \hat{c}_q)$ using the similarity transformation $\mathcal{C}_q$ in (4).

Step 2) Solve the Lyapunov (5) to determine the $n$th-order GIRG $W_{q,n}$.

Step 3) Partition the $(r+1)$th-order original GIRG as (12) and solve (13) to obtain an $r$th-degree characteristic polynomial.

Step 4) Form the reduced realization $(\hat{\hat{A}}_q, \hat{\hat{b}}_q, \hat{\hat{c}}_q)$: the state matrix $\hat{\hat{A}}_q$ follows from Step 3) with a structure as in (6) and $\{\hat{\hat{b}}_q, \hat{\hat{c}}_q\}$ matches the first $r$ entries of $\{\hat{b}_q, \hat{c}_q\}$.

The condition to preserve the initial stability is given by the next theorem.

*Theorem 3.1:* Let $(\hat{\hat{A}}_q, \hat{\hat{b}}_q, \hat{\hat{c}}_q)$ be the $r$th-order ROM of any asymptotically stable initial system $(A, b, c)$ derived using our GIRG-based algorithm: $\tilde{p}_q(z)$ cannot have any zeros outside the unit circle. Furthermore, provided $\{\hat{\hat{A}}_q, \hat{\hat{c}}_q\}$ is observable, the ROM is asymptotically stable.

*Proof:* From Theorem 2.1, it is known that the $n$th-order original GIRG $W_{q,n}$ is the solution to the Lyapunov equation $W_{q,n} - \hat{A}^T W_{q,n} \hat{A} = \hat{Q}_q$ with $\hat{Q}_q = \hat{c}_q^T \hat{c}_q$, where $(\hat{A}, \hat{b}_q, \hat{c}_q)$ is obtained from any realization $(A, b, c)$ using the similarity transformation $\mathcal{C}_q$ defined in (4). It is seen that the $r$th-order original GIRG solves the following equation:

$$W_{q,r} - \hat{\hat{A}}_q^T W_{q,r} \hat{\hat{A}}_q = \hat{Q}_q(1:r, 1:r) + \hat{\hat{Q}}_q^+ \tag{15}$$

with

$$\hat{\hat{Q}}_q^+ = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha \end{bmatrix}$$

$$\alpha = w_{q,r+1} - W_{q,r+1}(r+1:r+1, 1:r)\bar{\boldsymbol{a}}_q \tag{16}$$

where the subscripting notation $M(i:j, i':j')$ stands for the submatrix with rows $i \cdots j$ and columns $i' \cdots j'$ of matrix $M$.

$\hat{Q}_q$ is a positive-semidefinite matrix. Therefore, $\hat{Q}_q(1:r, 1:r)$ is positive semidefinite.

To get the required result, we shall now show that $\hat{\hat{Q}}_q^+$ is also a positive-semidefinite matrix.

It is seen from (13) that the following equation holds:

$$W_{q,r+1}(r+1:r+1, 1:r)\bar{\boldsymbol{a}}_q = \langle \tilde{w}_{q+r}, w_{q+r} \rangle \tag{17}$$

where $\tilde{w}_{q+r}[k]$ is the $l_2$-optimal approximation of the function $w_{q+r}[k]$ with the set $\{w_{q+i-1}[k]\}$:

$$\tilde{w}_{q+r}[k] = -\sum_{i=1}^{r} \bar{a}_i w_{q+r-i}[k]. \tag{18}$$

Hence, we have the following expression for the last diagonal entry of $\hat{\hat{Q}}_q^+$:

$$\alpha = \langle w_{q+r}, w_{q+r} \rangle - \langle \tilde{w}_{q+r}, w_{q+r} \rangle. \tag{19}$$

From the *orthogonality principle,* we know that the error $e[k] \hat{=} w_{q+r}[k] - \tilde{w}_{q+r}[k]$ is orthogonal to the approximating functions: $\langle e, w_{q+r-i} \rangle = 0$, $i = 1, \cdots, r$, which implies that the second scalar product in (19) is the energy of the approximate function

$$\langle \tilde{w}_{q+r}, w_{q+r} \rangle = \langle \tilde{w}_{q+r}, \tilde{w}_{q+r} \rangle. \tag{20}$$

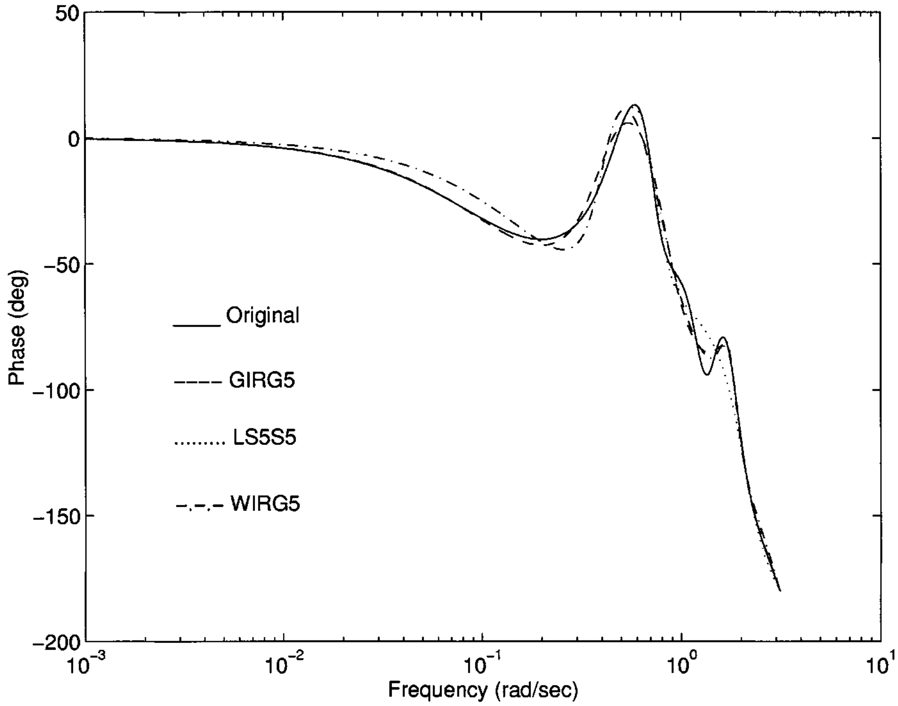


Fig. 2. Bode plots (phase).

TABLE I
ERRORS FOR THE REDUCED ORDER MODELS

| *Models* | Impulse Error (%) | Step Error | DC-Gain Error (%) |
|---|---|---|---|
| *GIRG5* | 1.27 | 0.1139 | 0 |
| *BR5* | 1.19 | $\infty$ | 6.25 |
| *WIRG5* | 1.13 | $\infty$ | 12.69 |
| *LS5S5* | 2.36 | 0.0691 | 0 |

A known property derived from the orthogonality principle is that the following inequality holds:

$$\langle \tilde{w}_{q+r},\, \tilde{w}_{q+r}\rangle \;<\; \langle w_{q+r},\, w_{q+r}\rangle . \tag{21}$$

Hence, it is seen from (16) that the only nonzero entry of $\hat{\hat{Q}}_q^+$ is positive, which implies that the right-hand side of the Lyapunov (15) is a positive-semidefinite matrix. Then, the characteristic polynomial of $\hat{\hat{A}}_q$ cannot have any zeros outside the unit circle, and, moreover, if $\{\hat{\hat{A}}_q,\, \hat{\hat{c}}_q\}$ is observable, $\hat{\hat{A}}_q$ is asymptotically stable. ■

*Remark 2:* Proof of Theorem 3.1 reveals an important difference between the discrete case and the continuous case considered in [15], in which Gramians are defined using operators $\int_{\infty}^{t}$ and/or $d/dt$. Here, the right-hand side of the Lyapunov (15) that solves the $r$th-order principal submatrice of the original GIRG is not only composed from the parameters $t_i$ or $m_i'$ of the original realization in (7), because of the additional term $\hat{\hat{Q}}_q^+$.

The controllability of the obtained ROM is now established in the following theorem.

*Theorem 3.2:* Provided that $\tilde{p}_q(1) \neq 0$, the $r$th-order ROM $(\hat{\hat{A}}_q,\, \hat{\hat{b}}_q,\, \hat{\hat{c}}_q)$ is controllable.

*Proof:* Let $\tilde{p}_q(z)$ be the characteristic polynomial for the state matrix $\hat{\hat{A}}_q$ and $\overline{\overline{p}}_q(z) = \sum_{i=0}^{r} \overline{\overline{a}}_i z^{r-i} = \tilde{p}_q(z+1)$. Because $\hat{\hat{A}}_q$ has the same structure as in (6), it is clear that $(\hat{\hat{A}}_q - I)$ is a companion matrix. Now, it is seen that

$$\det\left[\mathcal{C}\left\{\left(\hat{\hat{A}}_q - I\right),\, \hat{\hat{b}}_q\right\}\right] = \begin{cases} \overline{\overline{a}}_r^{\,-q}, & \text{if } r \text{ odd} \\ (-1)^{-q+r+1}\overline{\overline{a}}_r^{\,-q}, & \text{if } r \text{ even} \end{cases}$$

where $\mathcal{C}\{\cdot\}$ denotes the standard controllability matrix. Because $\mathcal{C}\{(\hat{\hat{A}}_q - I),\, \hat{\hat{b}}_q\}$ and $\mathcal{C}\{\hat{\hat{A}}_q,\, \hat{\hat{b}}_q\}$ have the same rank, the controllability matrix for our ROM realization is of full rank provided that $\overline{\overline{a}}_r \neq 0$. Noting that $\overline{\overline{a}}_r = \tilde{p}_q(1)$ achieves the proof.

Finally, assuming $\{\hat{\hat{A}}_q,\, \hat{\hat{c}}_q\}$ is observable yields the asymptotic stability (see Theorem 3.1), which implies $\tilde{p}_q(1) \neq 0$ and therefore controllability; in this case, the ROM is minimal. ■

*Remarks:*

1) As the ROM matches the $i$th Markov-type parameters $m_i'$ in (8), it also matches the $i$th Markov parameters.
2) It is well known that the high-frequency behavior is related to Markov parameters and the low-frequency one is related to time moments. Therefore, a GIRG with $q \sim 0$ is expected to give a better approximation at high frequencies than a GIRG with $q \sim -r+1$. Note that the original DC-gain is preserved for $q \neq 0$.
3) The present algorithm is easy to implement using a standard numerical software (e.g., the *MATLAB* script is ten lines long and available upon request to the authors).

## IV. EXAMPLE

Numerous examples have been studied in [17] to verify the validity of previous results.

Consider now the seventh-order transfer function of a supersonic jet engine inlet proposed by Lalonde in [18] as shown in (22a), at the top of the next page.

With this model, the characteristics of any order reduction technique is clearly highlighted by comparing the ROM frequency response with the original response, which is characterized by peaks at distinct frequencies.

Using our GIRG-based technique of the previous section with $q = -1$, we get the following five-order model (GIRG5) as shown in (22b), at the top of the next page.

$$h(z) = \frac{2.0434z^6 - 4.9825z^5 + 6.57z^4 - 5.8189z^3 + 3.636z^2 - 1.4105z + 0.2997}{z^7 - 2.46z^6 + 3.433z^5 - 3.333z^4 + 2.5460z^3 - 1.584z^2 + 0.7478z - 0.2520} \quad (22a)$$

$$\tilde{h}(z) = \frac{2.0434z^4 - 3.0842z^3 + 2.1696z^2 - 1.4130z + 0.7100}{z^5 - 1.5310z^4 + 1.2594z^3 - 0.9770z^2 + 0.6962z - 0.3241} \quad (22b)$$

To measure the approximations, consider the error criteria $\{Q = \|e\|_2^2, e[k] = y[k] - \tilde{y}[k]\}$, where $y[k]$ and $\tilde{y}[k]$ are the responses of the original and reduced-order model, respectively. For impulse responses, the criteria is usually normalized: $Q' = Q/\|h\|_2^2$.

Table I compares GIRG5 with models derived through balanced realization (BR5, see [13]), WIRG5 (see [2]) and least-squares with scaling (LS5S5, see [18]).

Models BR5 and WIRG5 give the best approximations from the point of view of the impulse response, but their step responses are not acceptable. Model LS5S5 provides a reasonable impulse response and a close approximation of the original step response. Model derived by GIRG exhibits good behavior on both impulse and step responses (the DC-gain is retained). The Bode plots of the original and reduced-order models (WIRG5, LS5S5, GIRG5) are shown in Figs. 1 and 2. Small reduction errors are obtained with model GIRG5 at high frequencies and low frequencies, as well as middle frequencies.

## V. Conclusion

A new impulse response Gramian has been introduced for linear, time-invariant, asymptotically stable, discrete SISO systems. It is easily obtained by solving a Lyapunov equation for a particular realization, and it is connected to standard Gramians. It has been further shown that it contains information about the characteristic polynomial. A model reduction method based on these properties has been proposed. The $r$th-order ROM is chosen in a set of $r$ solutions: the poles are first computed through a minimization of a $l_2$ error criteria and then we match some Markov parameters or time moments. This ROM cannot have any poles outside the unit circle and is asymptotically stable and minimal provided it is observable. This method can ensure a close approximation for a given frequency range. As shown by the numerical example, the proposed solution compares well with those obtained with other techniques.

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# Closed-Form Control Laws for Linear Time-Varying Systems

Ping Lu

***Abstract*—Closed-form control laws are developed for continuous, linear, time-varying (LTV) systems based on approximate solutions to a receding-horizon control problem. These control laws can be derived in the first- or higher order closed forms. Once obtained, the control laws need no explicit gain-scheduling or online integrations to implement. The notion of practical stability is used, and practical or uniform asymptotic stability of the closed-loop system, depending on conditions imposed on the system, is established.**

***Index Terms*—Linear time-varying systems, optimal control, quadratic programming, receding-horizon control.**

## I. Introduction

Relatively few methods for controller design have been available to stabilize a linear, time-varying (LTV) system. The contrast is par-

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